Taylor and Maclaurin series (11.10)

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This is the Taylor series/expansion of f at a

Maclaurin series: this with $a=0$

$$\underbrace{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$f(x) - T_n(x)$

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and if for all x in $(a-R, a+R)$

$$\lim_{n\to\infty} R_n(x) = 0,$$

~~for $|x-a|<R$,~~ then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

$(a-R, a+R)$

So: radius of convergence $\geq R$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \binom{x}{k} = \frac{x}{k}\cdot\frac{x-1}{k-1}\cdot\frac{x-2}{k-2}\cdots\frac{(\;)}{1}$$

$a-d \leq x \leq a+d$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leq M_n$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M_n}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

This lets us compute approximations of $f(x)$ on an interval. Can help in applying Theorem 8

(Useful fact: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

17 The Binomial Series If k is any real number and $|x| < 1$, then

Newton:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$1-x\,\overline{)\,1\,}\quad = 1 + x + x^2 + x^3 + \cdots$$

$$\begin{array}{l} 1 \\ \underline{1 - x} \\ \quad x \\ \quad \underline{x - x^2} \\ \qquad x^2 \\ \qquad \underline{x^2 - x^3} \\ \qquad\qquad x^3 \end{array}$$

$$1+x^2\,\overline{)\,1\,}\quad = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\begin{array}{l} 1 \\ \underline{1 + x^2} \\ \quad -x^2 \\ \quad \underline{-x^2 - x^4} \\ \qquad\quad x^4 \end{array}$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

$$\ln(1+x) = \int_1^x \frac{1}{t}\,dt$$

ex $$\lim_{x\to 0} \frac{\sin(x) - \left(x - \frac{x^3}{3!}\right)}{x^5} = \lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - \left(\cancel{x} - \cancel{\frac{x^3}{3!}}\right)}{x^5}$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots\right) = \boxed{\frac{1}{5!}}$$

$f(x) = \sqrt{x} = x^{1/2}$ centered at 2

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

$f''(x) = \frac{-1}{2 \cdot 2} x^{-\frac{3}{2}}$

$f'''(x) = \frac{1 \cdot 3}{2 \cdot 2 \cdot 2} x^{-5/2}$

$f''''(x) = \frac{-1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 2} x^{-7/2}$

$f'''''(x) = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} x^{-9/2}$

n	$f^{(n)}(2)$		
0	$2^{1/2}$		
1	$\frac{1}{2 \cdot 2^{1/2}}$	$=$	$\frac{\sqrt{2}}{2^2}$
2	$\frac{-1}{2^2 \cdot 2^{3/2}}$	$=$	$\frac{-\sqrt{2}}{2^4}$
3	$\frac{1 \cdot 3}{2^3 \cdot 2^{5/2}}$	$=$	$\frac{1 \cdot 3 \cdot \sqrt{2}}{2^6}$
4	$\frac{-1 \cdot 3 \cdot 5}{2^4 \cdot 2^{7/2}}$	$=$	$\frac{-1 \cdot 3 \cdot 5 \sqrt{2}}{2^8}$
5	$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 \cdot 2^{9/2}}$	$=$	$\frac{1 \cdot 3 \cdot 5 \cdot 7 \sqrt{2}}{2^{10}}$

$$f^{(n)}(2) = \frac{(-1)^{n+1} \, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{4^n} \sqrt{2}$$

$$\sqrt{x} = 2^{1/2} + \frac{\sqrt{2}}{4}(x-2) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{4^n} \sqrt{2} \, \frac{(x-2)^n}{n!}$$

$(1+x)^k$'s series matches for $-1<x<1$

$$\sqrt{x} = (x)^{1/2} = (2+(x-2))^{1/2} = \sqrt{2}\left(1+\frac{(x-2)}{2}\right)^{1/2} = \sqrt{2}\left(\sum_{n=0}^{\infty}\binom{1/2}{n}\left(\frac{x-2}{2}\right)^n\right)$$

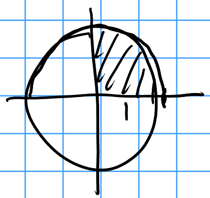
valid for $-1 < \frac{x-2}{2} < 1$

$\iff -2 < x-2 < 2$

$\iff \boxed{0 < x < 4}$

$$\sqrt{1-x^2} = (1-x^2)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-x^2)^n$$

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$



$$\pi = 4\int_0^1 \sqrt{1-x^2}\,dx = 4\int_0^1 \sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n x^{2n}\,dx$$

$$= 4\sum_{n=0}^{\infty}\int_0^1 \binom{1/2}{n}(-1)^n x^{2n}\,dx = 4\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n \frac{1}{2n+1}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$f(x) = \arctan(x^2)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}$$

$$f(x) = \arctan(x-1) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n+1}}{2n+1}$$

$$f(x) = \arctan(1-x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-x^2)^{2n+1}}{2n+1}$$

$$(1-x^2)^{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k x^{2k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{2n+1} \frac{(-1)^n}{2n+1} \cdot \binom{2n+1}{k} (-1)^k x^{2k}$$

$$f(x) = x^2 \ln(1+2x)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

$$= x^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (2x)^n = x^2 \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^{n+2}}{n} = \sum_{m=3}^{\infty} \frac{(-1)^m 2^{m-2} x^m}{m-2}$$

$$m = n+2$$

$$(-1)^n = (-1)^{m-2} = (-1)^m \cancel{(-1)^2}$$